Running coupling constants, Newtonian potential, and nonlocalities in the effective action

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We consider a quantum scalar field on an arbitrary gravitational background. We obtain the effective in-in equations for the gravitational fields using a covariant and nonlocal approximation for the effective action proposed by Vilkovisky and collaborators. From these equations, we compute the quantum corrections to the Newtonian potential. We find logarithmic corrections which we identify as the running of the gravitational constants. This running coincides with the renormalization group prediction only for minimal and conformal coupling.

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I. INTRODUCTION

The effective action (EA) is a useful tool for analyzing the quantum corrections to the classical dynamics in quantum field theory. In particular, the effective equations derived from it should be the starting point to investigate many interesting problems such as the influence of quantum matter fields on the behavior of gravitational fields, both in cosmology and black hole physics.

The EA is a very complicated object, even in the one-loop approximation, and it is necessary to develop approximation techniques in order to evaluate it. A widely used approximation is the Schwinger-DeWitt (SDW) expansion [1], which consists basically of an expansion in derivatives of the background fields. This expansion is useful in situations where the background fields are slowly varying with respect to the mass of the quantum fluctuations and for the analysis of the renormalizability of the theory. However, many interesting physical effects are lost in this approximation. Alternatively, one can consider a situation where the background fields are weak but rapidly varying. In this case, it is possible to expand the EA in powers of the curvatures of the background fields. The resulting expansion has been recently investigated by Vilkovisky and collaborators [2–4], and it is in general a nonlocal object.

On the other hand, there is a simple and intuitive way of taking into account, at least partially, the quantum effects. In quantum field theory, parameters such as masses and coupling constants are not constants but scale-dependent quantities. This is due to vacuum polarization effects and the scale dependence is dictated by the renormalization group equations. One can use this fact to construct a "Wilsonian" effective action [5], which is basically the classical action in which the parameters have been replaced by their running counterparts. An argument of this type has been recently proposed to explain the dark matter problem [6]: because of quantum effects the Newtonian potential should be modified according to

$$V(r) = -\frac{G(\mu = \frac{1}{r})M}{r} \tag{1}$$

where $G(\mu)$ is the solution to the renormalization group equations in a renormalizable theory of gravity with R^2 terms in the Lagrangian [7]. As the theory is asymptotically free, $G(r)$ is an increasing function of r, and this may explain at least part of the "missing" mass. The running of G may also induce interesting cosmological and astrophysical effects [8].

It is the aim of this work to analyze the relationship between the nonlocal approach to the EA proposed by Vilkovisky and collaborators, the renormalization group, and the "Wilsonian" effective action. In Ref. [9], it was shown that the existence of nonlocal terms in the effective action is linked to the short distance behavior of the theory and to the renormalization group. The analysis was done in a noncovariant weak-field approximation, at the level of the in-out effective action. Here we will extend that analysis: we will be using a covariant effective action, we will work at the level of the in-in (see below) semiclassical equations of motion, and we will see explicitly the running behavior of the gravitational constants in the Newtonian potential. For simplicity, we will consider a toy model in which we quantize a scalar field on a classical gravitational field. We will not include the graviton loop in our calculations.

We would like to stress that our interest here is not to look for measurable corrections to the Newtonian potential. Indeed, we know *a priori* that these corrections are extremely small in the toy model considered. What we are going to discuss is how to derive from "first principles" (i.e., the EA) the scale dependence of G in the gravitational potential.

The paper is organized as follows. As a warm up, in the next section we will obtain the running behavior of the electric charge in QED starting from the nonlocal EA. In Sec. III we will consider a free quantum scalar field of mass m on an arbitrary gravitational background. We will obtain the nonlocal version of the effective action in powers of $-\frac{m^2}{\Box}$, and the nonlocal effective equations. In Sec. IV we will compute the quantum corrections to the Newtonian potential and compare the results with the ones obtained from the "Wilsonian" approach. Section V contains the conclusions of our work.

II. QUANTUM ELECTRODYNAMICS

Because of vacuum polarization, the electrostatic interaction potential between point charges is modified to [10]

$$\begin{aligned} V_{\rm int}(r) &= \frac{e^2(r)}{4\pi} \\ &= \frac{e^2}{4\pi r}\left[1+\frac{e^2}{6\pi^2}\int_1^\infty du\, e^{-2mru}\left(1+\frac{1}{2u^2}\right)\right. \\ &\quad \left.\times\frac{\sqrt{u^2-1}}{u^2}+O(e^4)\right]. \end{aligned} \tag{2}$$

In the short distance limit ($mr \ll 1$) we have

$$e(r) = e\left[1-\frac{e^2}{12\pi^2}\ln\frac{r}{r_0}+O(e^4)\right] \tag{3}$$

where r_0 is defined by $-\ln mr_0 = 2\gamma+\frac{5}{3}$.

On the other hand, the solution to the renormalization group equation gives the following running for the electric charge:

$$e(\mu) = e(\mu_0)\left[1-\frac{e^2(\mu_0)}{12\pi^2}\ln\frac{\mu_0}{\mu}+O(e^4)\right]. \tag{4}$$

As can be easily seen from Eq. (3), in the short distance limit the electrostatic interaction potential is just the usual $\frac{e^2}{4\pi r}$ in which the electric charge has been replaced by its running counterpart, Eq. (4), with the additional rule that the mass scale μ is replaced by the inverse of the distance r and the mass scale reference μ_0 is set at r_0^{-1}. Therefore the "Wilsonian" argument gives the correct answer for the electrostatic potential in the short distance limit.

We will derive here these old and well-known results using the EA formalism, since this exercise will be a useful guide to the more complex calculation presented in Secs. III and IV. For simplicity we will quantize only the fermion field (with no classical component), keeping the electromagnetic field as a classical background.

The classical action for QED in Euclidean space is given by

$$S_c = \int d^4x[\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(\partial\!\!\!/+ie\,A\!\!\!/+im)\psi]. \tag{5}$$

In the weak field approximation, the one-loop effective action obtained after integrating out the fermions is given by

$$\begin{aligned} S_{\rm eff} &= S_c+\frac{e^2}{2}\int d^4x d^4x' A^\mu(x)\Pi_{\mu\nu}(x,x') \\ &\quad \times A^\nu(x')+O(A^4) \end{aligned} \tag{6}$$

where $\Pi_{\mu\nu}$ is the usual vacuum polarization tensor for QED. The renormalized effective action is

$$S_{\rm eff} = \frac{1}{4}\int d^4x F_{\mu\nu}\left[1-\frac{e^2}{\pi^2}F(\Box)\right]F^{\mu\nu}+O(A^4) \tag{7}$$

where

$$F(\Box) = \frac{1}{8}\int_0^1 (1-t^2)\ln\left[\frac{m^2-\frac{1}{4}(1-t^2)\Box}{\mu^2}\right]. \tag{8}$$

The modified Maxwell equations that derive from Eq. (7) are

$$\left[1-\frac{e^2}{\pi^2}F(\Box)\right]\partial_\mu F^{\mu\nu} = J^\nu_{\rm clas} \tag{9}$$

where we included a classical source $J^\nu_{\rm clas}$.

The form factor $F(\Box)$ admits the following integral representation in terms of the massive Euclidean propagator $(M^2-\Box)^{-1}$:

$$\begin{aligned} F(\Box) &= \frac{1}{8}\int_0^1 dt(1-t^2)\left[\ln\frac{(1-t^2)}{4}\right. \\ &\quad \left.+\int_0^\infty dz\left(\frac{1}{z+\mu^2}-\frac{1}{z+\frac{4m^2}{(1-t^2)}-\Box}\right)\right]. \end{aligned} \tag{10}$$

Therefore, we can regard $F(\Box)$ as a two-point function whose action on a test function $f(x)$ is given by

$$F(\Box)f(x) = \int d^4x' F(\Box)(x,x')f(x'). \tag{11}$$

All these equations are valid in Euclidean space. To get the Minkowski version of them, one should replace the Euclidean propagator by the Feynman one. However, the equations thus obtained are neither real nor causal since the effective action gives in-out matrix elements instead of expectation values, making the interpretation of the equations awkward. Alternatively, one can use the close time path (CTP) [11] formalism to construct an in-in effective action that produces real and causal field equations for in-in expectation values [12].

The CTP formalism involves a doubling of the degrees of freedom and a generalization of the Feynman rules that includes both the Feynman and Dyson propagators, as well as the two-point Wightman functions (these functions carry the information about the quantum state of the system). However, if one is interested in the in-in effective equations for the standard in-vacuum state, this complication can be avoided. Indeed, in this situation it can be shown that the in-in version of the equations is obtained by replacing the Euclidean propagator by the retarded one [3] in the integral representation of the form factor equation (10). Alternatively, the in-in form factor can be obtained in the in-out formalism by taking twice the real and causal part of the in-out form factor [13]. We will denote the in-in form factor thus obtained by $F_{\rm in}(\Box)$.

In particular, if the test function is time independent,

$$F_{\rm in}(\Box)f(\mathbf{x}) = F(\nabla^2)f(\mathbf{x}) = \int d^3x' \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}F(-\mathbf{k}^2)f(\mathbf{x}'), \tag{12}$$

because the time integral of the retarded propagator coincides with the Green function of the Laplacian.

In the short-distance limit $m^2 \ll -\Box$ the Euclidean field equation reduces to

$$\left[1 - \frac{e^2}{12\pi^2}\ln\left(-\frac{\Box}{\mu^2}\right)\right]\partial_\mu F^{\mu\nu} = J^\nu. \tag{13}$$

One often encounters the distribution $G(-\frac{\Box}{\mu^2}) = \ln(-\frac{\Box}{\mu^2})$, which will play a central role in what follows. The action of the in-in counterpart of $G(-\frac{\Box}{\mu^2})$ on a test function $f(x)$ is given by (see Refs. [13–15])

$$\begin{aligned} G_{\rm in}\left(-\frac{\Box}{\mu^2}\right)f(x) &= \frac{2}{\pi}\int d^4x'\theta(x^0 - x^{0\prime})\delta'((x-x')^2)f(x') \\ &= \frac{1}{2\pi}\int_0^\infty du \int_0^{4\pi} d\Omega \left[\ln(\mu u)\frac{\partial f}{\partial u}\bigg|_{v=0} \right. \\ &\quad \left. -\frac{1}{2}\frac{\partial f}{\partial v}\bigg|_{v=0}\right] \end{aligned} \tag{14}$$

where u and v are respectively the standard retarded and advanced coordinates with origin at the point x. When the test function is time independent, Eq. (14) reduces to

$$G_{\rm in}\left(-\frac{\Box}{\mu^2}\right)f(\mathbf{x}) = G\left(-\frac{\nabla^2}{\mu^2}\right)f(\mathbf{x}), \tag{15}$$

as expected from Eq. (12).

We are now ready to compute the modifications to the electrostatic potential. Taking as a classical source a static point charge, the modified Gauss law reads

$$\nabla\cdot\mathbf{E} - \frac{e^2}{12\pi^2}G\left(-\frac{\nabla^2}{\mu^2}\right)\nabla\cdot\mathbf{E} = e\delta^3(\mathbf{x}). \tag{16}$$

The solution for the electric field is spherically symmetric $\mathbf{E} = E(r)\hat{\mathbf{r}}$ and we shall find it perturbatively in powers of e^2:

$$\mathbf{E} = \mathbf{E}^{(0)} + \mathbf{E}^{(1)}. \tag{17}$$

The leading order term is the classical contribution

$$\nabla\cdot\mathbf{E}^{(0)} = e\delta^3(\mathbf{x}) \Longrightarrow E^{(0)}(r) = \frac{e}{4\pi r^2} \tag{18}$$

and the first quantum correction is given by

$$\nabla\cdot\mathbf{E}^{(1)} = \frac{e^2}{12\pi^2}G\left(-\frac{\nabla^2}{\mu^2}\right)\nabla\cdot\mathbf{E}^{(0)}. \tag{19}$$

Therefore, we have to evaluate the action of $G(-\frac{\nabla^2}{\mu^2})$ on the δ function. Using Eqs. (12) and (15) we readily obtain

$$G\left(-\frac{\nabla^2}{\mu^2}\right)\delta^3(\mathbf{x}) = -\frac{1}{2\pi r^3} - \ln\mu^2\delta^3(\mathbf{x}) \tag{20}$$

where the last term gives a μ-dependent correction to the classical solution that will be absorbed into the classical source. The quantum correction is

$$E^{(1)}(r) = E^{(1)}(r_0)\frac{r_0^2}{r^2} - \frac{e^3}{24\pi^3 r^2}\ln\left(\frac{r}{r_0}\right) \tag{21}$$

where r_0 is an arbitrary reference radius. Integrating Eqs. (18) and (21) and multiplying by the charge e we get the electrostatic interaction potential

$$V_{\rm int}(r) = \frac{e^2}{4\pi r}\left[1 - \frac{e^2}{6\pi^2}\ln\left(\frac{r}{r_0}\right) + O(e^4)\right]. \tag{22}$$

From this equation the running behavior of the electric charge follows, and it coincides with that of Eq. (3).

III. NONLOCAL EFFECTIVE EQUATIONS FOR THE GRAVITATIONAL FIELD

Let us now consider a quantum scalar field on a gravitational background. The Euclidean action for the theory is

$$S = S_{\rm grav} + S_{\rm matter} \tag{23}$$

where

$$S_{\rm grav} = -\int d^4x\sqrt{g}\left[\frac{1}{16\pi G}(R - 2\Lambda) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right] \tag{24}$$

and

$$S_{\rm matter} = \frac{1}{2}\int d^4x\sqrt{g}[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2 + \xi R\phi^2]. \tag{25}$$

We have included terms quadratic in the curvature since in any case they will appear in the renormalization procedure. As we will use ζ-function regularization, the constants G, Λ, α and β are finite (and dependent on a mass scale μ).

The effective action for the classical gravitational field can be obtained by integrating out the quantum scalar field. Formally the result is

$$S_{\rm eff} = S_{\rm grav} + \frac{1}{2}\ln\det\left[\frac{-\Box + m^2 + \xi R}{\mu^2}\right] \stackrel{\rm def}{=} S_{\rm grav} + \Gamma \tag{26}$$

where μ is an arbitrary mass scale.

The evaluation of the above determinant in a general background is a very complicated task. Let us denote by $\mathcal{R}$ either the Riemann tensor or any of its contractions with the metric. When the gravitational field is slowly varying, i.e., when $\nabla^n \mathcal{R}^m \ll m^{n+2m}$, one can use the SDW technique to get [16]

$$\Gamma = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\left[\frac{1}{2}m^4 \ln\left(\frac{m^2}{\mu^2}\right) - m^2 a_1(x)\ln\left(\frac{m^2}{\mu^2}\right) + a_2(x)\ln\left(\frac{m^2}{\mu^2}\right) + \frac{1}{2}\sum_{j\geq 3} a_j(x)(m^2)^{-j-4}(j-3)!\right], \tag{27}$$

where we have omitted μ-independent terms which redefine the classical constants. The functions $a_j(x)$ are the coincidence limit of the SDW coefficients, given by

$$\begin{aligned} a_0(x) &= 1, \\ a_1(x) &= (\tfrac{1}{6}-\xi)R, \qquad (28)\\ a_2(x) &= \tfrac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \tfrac{1}{180}R_{\mu\nu}R^{\mu\nu} \\ &\quad -\tfrac{1}{6}(\tfrac{1}{5}-\xi)\,\Box R + \tfrac{1}{2}\left(\tfrac{1}{6}-\xi\right)^2 R^2, \\ &\;\;\vdots \\ a_n(x) &= \nabla^{2n-2}\mathcal{R} + \mathcal{R}\nabla^{2n-4}\mathcal{R} + \cdots + \nabla\nabla\mathcal{R}^{n-1} + \mathcal{R}^n. \qquad (29) \end{aligned}$$

The last line shows schematically the coincident limit of $a_n(x)$.

From the SDW expansion it is easy to derive the scaling of the gravitational constants. The effective action should not depend on the scale μ. As a consequence, taking μ derivatives in Eq. (26) we find

$$\mu\frac{dG}{d\mu} = \frac{G^2m^2}{\pi}\left(\xi - \tfrac{1}{6}\right), \tag{30}$$

$$\mu\frac{d\alpha}{d\mu} = -\frac{1}{32\pi^2}\left[\left(\tfrac{1}{6}-\xi\right)^2 - \frac{1}{90}\right], \tag{31}$$

$$\mu\frac{d\beta}{d\mu} = -\frac{1}{960\pi^2}, \tag{32}$$

$$\mu\frac{d}{d\mu}\frac{\Lambda}{G} = \frac{m^4}{4\pi}, \tag{33}$$

which is the usual running for the gravitational constants. We can use Eq. (30) to construct a "Wilsonian" gravitational potential. The scaling of G is given by

$$G(\mu) = G_0\left(1 + \frac{m^2G_0}{\pi}(\xi - \tfrac{1}{6})\ln\frac{\mu}{\mu_0}\right) \tag{34}$$

so the Wilsonian potential is

$$V(r) = -\frac{G_0M}{r}\left(1 - \frac{m^2G_0}{\pi}(\xi - \tfrac{1}{6})\ln\frac{r}{r_0}\right). \tag{35}$$

In the next section we will see if it is possible to derive this potential from the EA.

The SDW expansion is not useful for the analysis of the short distance behavior of the theory. As we have seen in the previous section, one should consider weak but rapidly varying background fields. Assuming that $\nabla\nabla\mathcal{R} \gg \mathcal{R}^2$, one may try to sum up in Eq. (27) all the terms which contain a given power of the curvature. This rather complicated calculation has been performed by Avramidy in Ref. [4]. See also Refs. [3] for the massless case. The result, up to second order in the curvature, is

$$\Gamma = \Gamma_{\rm local} + \Gamma_{\rm nonloc} \tag{36}$$

where

$$\begin{aligned} \Gamma_{\rm local} &= \frac{1}{64\pi^2}\int d^4x\sqrt{g}\left\{m^4\left[-\frac{3}{2} + \ln\left(\frac{m^2}{\mu^2}\right)\right] + 2m^2\left[-1+\ln\left(\frac{m^2}{\mu^2}\right)\right]\left(\xi - \frac{1}{6}\right)R\right\}, \\ \Gamma_{\rm nonloc} &= \frac{1}{32\pi^2}\int d^4x\sqrt{g}\left[RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu} + O(R^3)\right] \end{aligned} \tag{37}$$

and

$$\begin{aligned} F_1(\Box) &= \tfrac{1}{2}\int_0^1 dt\,\left[\xi^2 - \tfrac{1}{2}\xi(1-t^2) + \tfrac{1}{48}(3-6t^2-t^4)\right] \times \ln\left[\frac{m^2 - \frac{1}{4}(1-t^2)\Box}{\mu^2}\right], \\ F_2(\Box) &= \frac{1}{12}\int_0^1 dt\; t^4 \ln\left[\frac{m^2 - \frac{1}{4}(1-t^2)\Box}{\mu^2}\right]. \end{aligned} \tag{38}$$

Note the similarities between these form factors and the corresponding $F(\Box)$ that appears in QED [Eq. (8)].

From Eqs. (37) and (38), one can derive the effective gravitational field equations. As we are neglecting $O(\mathcal{R}^3)$ terms in the effective action, it makes no sense to retain $O(\mathcal{R}^2)$ terms in the equations of motion. Therefore, when doing the variation of the action, it is not necessary to take into account the $g_{\mu\nu}$ dependence of the form factors. Moreover, it is possible to commute the covariant derivatives acting on a curvature, i.e., $\nabla_\mu\nabla_\nu\mathcal{R} = \nabla_\nu\nabla_\mu\mathcal{R} + O(\mathcal{R}^2)$. After a straightforward calculation we find

$$\begin{aligned} &\left[-\frac{1}{8\pi G} + \frac{m^2}{16\pi^2}(\xi - \tfrac{1}{6})\left(-1 + \ln\frac{m^2}{\mu^2}\right)\right](R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}) \\ &-g_{\mu\nu}\left[\frac{\Lambda}{8\pi G} + \frac{m^4}{64\pi^2}\left(-\frac{3}{2} + \ln\frac{m^2}{\mu^2}\right)\right] + \alpha H^{(1)}_{\mu\nu} + \beta H^{(2)}_{\mu\nu} \\ &= \frac{2}{\sqrt{g}}\frac{\delta\Gamma_{\rm nonloc}}{\delta g^{\mu\nu}} = \langle T_{\mu\nu}\rangle \end{aligned} \tag{39}$$

where

$$\begin{aligned} H^{(1)}_{\mu\nu} &= 4\nabla_\mu\nabla_\nu R - 4g_{\mu\nu}\,\Box R + O(R^2), \\ H^{(2)}_{\mu\nu} &= 2\nabla_\mu\nabla_\nu R - g_{\mu\nu}\,\Box R - 2\,\Box R_{\mu\nu} + O(R^2), \end{aligned} \tag{40}$$

and

$$\langle T_{\mu\nu}\rangle = \frac{1}{32\pi^2}\left[F_1(\Box)H^{(1)}_{\mu\nu} + F_2(\Box)H^{(2)}_{\mu\nu}\right]. \tag{41}$$

Up to here we made no assumptions about the mass m. In the large mass limit, $m^2\mathcal{R} \gg \nabla\nabla\mathcal{R}$ the SDW expansion equation (27) is recovered [up to $O(\mathcal{R}^3)$]. However, as in QED, we are interested in the opposite limit. Let us assume that the typical scale of variation of the gravitational field is much smaller than m^{-1}, that is, $m^2\mathcal{R} \ll \nabla\nabla\mathcal{R}$. In this situation, we can expand the functions $F_1(\Box)$ and $F_2(\Box)$ in powers of $-\frac{m^2}{\Box}$. The result is

$$F_1(\Box) = \left[-\frac{1}{1800} + \frac{5\xi}{18} - \xi^2 + \frac{1}{2}\left[(\xi-\tfrac{1}{6})^2 - \tfrac{1}{90}\right]\ln\left(-\frac{\Box}{\mu^2}\right)\right] + \left[\frac{4}{18} - \xi + \xi^2 + (\xi^2 - \tfrac{1}{12})\ln\left(-\frac{\Box}{m^2}\right)\right]\left(-\frac{m^2}{\Box}\right) + O\left(-\frac{m^2}{\Box}\right)^2 \tag{42}$$

and

$$F_2(\Box) = \left[-\frac{23}{450} + \frac{1}{60}\ln\left(-\frac{\Box}{\mu^2}\right)\right] + \left[-\frac{5}{18} + \frac{1}{6}\ln\left(-\frac{\Box}{m^2}\right)\right]\left(-\frac{m^2}{\Box}\right) + O\left(-\frac{m^2}{\Box}\right)^2. \tag{43}$$

(It is possible to obtain exact expressions for F_1 and F_2 in terms of elementary functions. However, we will not need these long expressions in what follows.) Inserting the expansions, Eqs. (42) and (43), into the effective equations, Eq. (39), we get

$$\begin{aligned}&\left[\alpha - \frac{1}{32\pi^2}\left(-\frac{1}{1800} + \frac{5\xi}{18} - \xi^2\right) - \frac{1}{64\pi^2}\left[(\xi-\tfrac{1}{6})^2 - \tfrac{1}{90}\right]\ln\left(-\frac{\Box}{\mu^2}\right)\right]H^{(1)}_{\mu\nu} + \left[\beta - \frac{1}{32\pi^2}\left\{-\frac{23}{450} + \frac{1}{60}\ln\left(-\frac{\Box}{\mu^2}\right)\right\}\right]H^{(2)}_{\mu\nu}\\ &+\left[-\frac{1}{8\pi G} + \frac{m^2}{16\pi^2}(\xi - \tfrac{1}{6})\left(-1 + \ln\frac{m^2}{\mu^2}\right)\right]\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\right) + \left[\frac{m^2}{32\pi^2}\left(\tfrac{4}{18} - \xi + \xi^2\right)\right]\frac{1}{\Box}H^{(1)}_{\mu\nu} - \left[\frac{5m^2}{576\pi^2}\right]\frac{1}{\Box}H^{(2)}_{\mu\nu}\\ &+\left[\frac{m^2\xi^2}{32\pi^2}\right]\ln\left(-\frac{\Box}{m^2}\right)\frac{1}{\Box}H^{(1)}_{\mu\nu} - \left[\frac{m^2}{384\pi^2}\right]\ln\left(-\frac{\Box}{m^2}\right)\frac{1}{\Box}(H^{(1)}_{\mu\nu} - 2H^{(2)}_{\mu\nu}) = -T^{\rm clas}_{\mu\nu},\end{aligned} \tag{44}$$

where we have set the scale μ so that the cosmological constant is zero and we included a classical source $T^{\rm clas}_{\mu\nu}$.

As with the SDW expansion equation (27), one can easily derive the μ dependence of the gravitational constants from these modified Einstein equations. Alternatively, as pointed out in Refs. [17] and [9], one should also see the running behavior by performing the rescaling $g_{\mu\nu} \to s^{-2}g_{\mu\nu}$ and looking at the large s limit. Since $\Box \to s^2\Box$ under this rescaling, the nonlocal terms proportional to $\ln\Box$ become relevant in this limit. From the terms independent of m in Eq. (44) we get

$$\alpha(s) = \alpha(s=1) - \frac{1}{32\pi^2}\left[(\xi - \tfrac{1}{6})^2 - \tfrac{1}{90}\right]\ln s, \tag{45}$$

$$\beta(s) = \beta(s=1) - \frac{1}{960\pi^2}\ln s. \tag{46}$$

It is worth noting that the scaling behavior for α and β obtained using both methods, Eqs. (31),(32) and (45),(46), is identical. As far as the Newton constant is concerned, we can obtain its running behavior only for $\xi = 0$. In this particular case, the terms proportional to m^2 in Eq. (44) have a logarithmic kernel that appears in the combination

$$-\frac{m^2}{384\pi^2}\ln\left(-\frac{\Box}{m^2}\right)\frac{1}{\Box}(H^{(1)}_{\mu\nu} - 2H^{(2)}_{\mu\nu}). \tag{47}$$

Up to the order we are working ($O(\mathcal{R}^2)$), the basic tensors $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, $H^{(1)}_{\mu\nu}$, and $H^{(2)}_{\mu\nu}$ are related by

$$H^{(1)}_{\mu\nu} - 2H^{(2)}_{\mu\nu} = 4\,\Box\left(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\right). \tag{48}$$

Therefore, it is natural to express Eq. (47) using this relation, which thus leads to an s scaling for G,

$$G(s) = G(s=1)\left(1 - \frac{G(s=1)m^2}{6\pi}\ln s\right) \tag{49}$$

that is identical to the μ scaling equation (34) for minimal coupling. For an arbitrary coupling we cannot get the s scaling for G, since the logarithmic kernel does not appear in the simple combination equation (47). In this case Eq. (48) makes the identification of the scaling behavior ambiguous.

We shall see in the next section how to obtain the running behavior for G from the Newtonian potential.

IV. QUANTUM CORRECTIONS TO THE NEWTONIAN POTENTIAL

As in QED, the vacuum polarization effects contained in $\langle T_{\mu\nu}\rangle$ induce modifications to the Newtonian potential. We will now evaluate these corrections. To begin with, we must obtain the in-in effective equations. To this end, we should express the form factors as integrals of the massive Euclidean propagator and replace it by the retarded propagator [see Eq. (10)]. However, when computing the Newtonian potential we will consider only time independent fields. Therefore, $F_{\rm in}(\Box) = F(\nabla^2)$ and the in-in equations are just the Euclidean equations with $\Box$ substituted by ∇^2.

In the static weak-field approximation we have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad | h_{\mu\nu} | \ll 1,$$
$$R = \frac{1}{2}\nabla^2 h, \tag{50}$$

where we assumed the Lorentz gauge conditions $(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h)_{;\nu} = 0$. For a point particle with $T_{\mu\nu} = \delta^0_\mu\delta^0_\nu M\delta^3(\mathbf{x})$, the trace of (linearized) Eq. (44) is

$$\left[\frac{1}{16\pi G}\nabla^2 - 2(3\alpha+\beta)\nabla^2\nabla^2 - \frac{1}{32\pi^2}\left\{\frac{19}{180} - \frac{5\xi}{3} + 6\xi^2 - 3(\xi - \tfrac{1}{6})^2 \ln\left(-\frac{\nabla^2}{\mu^2}\right)\right\}\nabla^2\nabla^2\right.$$
$$\left. - \frac{m^2}{16\pi^2}\left\{\frac{1}{2}(\xi - \tfrac{1}{6})\left(-1 + \ln\frac{m^2}{\mu^2}\right) + \frac{7}{18} - 3\xi + 3\xi^2 + 3\left(\xi^2 - \frac{1}{36}\right)\ln\left(-\frac{\nabla^2}{\mu^2}\right)\right\}\nabla^2\right]h + O(m^4) = -M\delta^3(\mathbf{x}). \tag{51}$$

For simplicity we shall compute only the trace h and not the complete $h_{\mu\nu}$, since this will simplify the calculations and will be enough for our purposes. In the limit $\alpha, \beta \to 0$, $-h$ is four times the Newtonian potential.

We shall solve Eq. (51) perturbatively:

$$h = h^{(0)} + h^{(1)}. \tag{52}$$

The classical contribution $h^{(0)}$ satisfies

$$(\nabla^2 - \sigma^{-2}\nabla^2\nabla^2)h^{(0)} = -16\pi G M\delta^3(\mathbf{x}) \tag{53}$$

where $\sigma^{-2} = 32\pi G(3\alpha + \beta)$. The time independent and spherically symmetric solution is [18]

$$h^{(0)} = \frac{4GM}{r}(1 - e^{-\sigma r}). \tag{54}$$

The first quantum correction satisfies

$$(\nabla^2 - \sigma^{-2}\nabla^2\nabla^2)h^{(1)} = H(\nabla^2)h^{(0)} \tag{55}$$

where

$$H(\nabla^2) = \frac{G}{2\pi}\left[\frac{19}{180} - \frac{5\xi}{3} + 6\xi^2 - 3(\xi - \tfrac{1}{6})^2 G\left(-\frac{\nabla^2}{\mu^2}\right)\right]\nabla^2\nabla^2 + \frac{Gm^2}{\pi}$$
$$\times\left[\frac{1}{2}(\xi - \tfrac{1}{6})\left(-1 + \ln\frac{m^2}{\mu^2}\right) + \frac{7}{18} - 3 + 3\xi^2 + 3(\xi^2 - \tfrac{1}{36})G\left(-\frac{\nabla^2}{m^2}\right)\right]\nabla^2. \tag{56}$$

We now find the solution to this equation. To begin with, we will consider the limit $\sigma r \to \infty$, since in this approximation it is easy to find such a solution. In this limit the classical potential becomes

$$h^{(0)} = 4GM\left(\frac{1}{r} + 4\pi\sigma^{-2}\delta^3(\mathbf{x})\right). \tag{57}$$

Using the action of the kernel $G(-\frac{\nabla^2}{\mu^2})$ on the δ function [Eq.(20)] we find

$$\frac{1}{64\pi G^2 M}(\nabla^2 - \sigma^{-2}\nabla^2\nabla^2)h^{(1)} = A\,\delta^3(\mathbf{x}) + B\,\nabla^2\delta^3(\mathbf{x}) + C\,\nabla^2\nabla^2\delta^3(\mathbf{x}) + \left[\frac{3m^2}{8\pi^2}(\xi^2 - \tfrac{1}{36})\right]\frac{1}{r^3}$$
$$+\left[-\frac{9}{8\pi^2}(\xi - \tfrac{1}{6})^2 + \frac{9\sigma^{-2}m^2}{4\pi^2}(\xi^2 - \tfrac{1}{36})\right]\frac{1}{r^5} + \left[\frac{45\sigma^{-2}}{2\pi^2}(\xi - \tfrac{1}{6})^2\right]\frac{1}{r^7} \tag{58}$$

where the coefficients A, B, and C depend on m, μ, and ξ. The solution to this equation is

$$h^{(1)} = -\frac{24G^2Mm^2}{\pi}(\xi^2 - \tfrac{1}{36})\frac{\ln\frac{r}{r_0}}{r} - \frac{12G^2M}{\pi}(\xi - \tfrac{1}{6})^2\frac{1}{r^3} - \frac{72G^2\sigma^{-2}M}{\pi}(\xi - \tfrac{1}{6})^2\frac{1}{r^5} + O(\sigma^{-4}) + \cdots. \tag{59}$$

The first, second, and third terms come from the sources r^{-3}, r^{-5}, and r^{-7}. The ellipsis denotes a term proportional to the classical solution $h^{(0)}$ as well as corrections at the origin, which are proportional to $\delta^3(\mathbf{x})$ and its derivatives. They all come from the sources proportional to A, B, and C in Eq. (58). We have not included them because our quantum corrections are not accurate near the origin. Indeed, we have derived the modified Einstein equations under the assumptions $\nabla\nabla\mathcal{R} \gg \mathcal{R}^2$ and $m^2\mathcal{R} \ll \nabla\nabla R$. Both conditions are satisfied for the $\frac{GM}{r}$ potential if $GM \ll r \ll m^{-1}$, so the origin $r = 0$ is excluded.

From Eq. (59) we see that there are two different types of terms in the quantum correction. The term containing the logarithm comes from the nonlocal terms proportional to $m^2 \ln(-\Box)$ in Eq. (44). It is qualitatively what we expected from "Wilsonian" arguments. However, the coefficient $\frac{24G^2Mm^2}{\pi}(\xi^2 - \frac{1}{36})$ is not exactly the same as the one derived from the renormalization group equation (34), unless $\xi = 0$ or $\xi = \frac{1}{6}$, i.e., minimal or conformal coupling. This is an important difference with respect to the QED calculation, and shows that the "Wilsonian" arguments are not always quantitatively correct. In addition to the running of G, we have found additional r^{-3} and r^{-5} corrections.

There are no terms in $h^{(1)}$ that we could associate to a running of the constants α and β. This is not surprising because such a running would imply terms of the form $\ln \frac{r}{r_0}\delta^3(\mathbf{x})$, which are ill defined. Moreover, we have already pointed out that our quantum corrections are not valid near the origin. Therefore, to see the running of these constants we shall evaluate the exact solution for h and then analyze the limit $\sigma r \to 0$ (to this end it is necessary to consider only the case $m^2 = 0$).

As σ is proportional to $|3\alpha + \beta|^{-\frac{1}{2}} l_{\text{Planck}}^{-1}$, the limit $\sigma r \to 0$ makes sense only for very large values of α and β. Otherwise the limit would apply only for r smaller than the Planck length l_{Planck}, where our semiclassical calculations are not valid. Therefore, in what follows we will assume that α and β take the maximum value compatible with experiments. This gives σ of order 10^{-4} cm^{-1} [18].

The calculations for the exact solution to Eq. (55) are presented in the Appendix. We quote here the main results. In order to solve the linearized equation of motion we have to evaluate the action of the kernel $G(-\frac{\nabla^2}{\mu^2})$ on the Yukawa potential

$$G\left(-\frac{\nabla^2}{\mu^2}\right)\frac{e^{-\sigma r}}{r} = \ln\left(\frac{\sigma^2}{\mu^2}\right)\frac{e^{-\sigma r}}{r} - \frac{e^{\sigma r}}{r}\text{Ei}(-\sigma r) - \frac{e^{-\sigma r}}{r}\text{Ei}(\sigma r) \quad (60)$$

where $\text{Ei}(x)$ is the exponential integral function. With the help of this formula we get the exact solution for $h^{(1)}$ (see the Appendix) and the limit $\sigma r \to 0$ can be taken. The solution reads

$$h^{(1)} = \frac{6G^2M\sigma^4}{\pi}(\xi - \tfrac{1}{6})^2\left[r\ln(\sigma r) + (\gamma - \tfrac{3}{2})r\right] + O(\sigma^6). \quad (61)$$

It is worth noting that the logarithmic term is exactly the one expected from the renormalization group scaling of $3\alpha + \beta$. Indeed, for small σr the classical potential becomes, up to a constant,

$$h^{(0)} \simeq -2MG\sigma^2 r. \quad (62)$$

Substituting in the above equation $G\sigma^2 = [32\pi(3\alpha + \beta)]^{-1}$ by its running counterpart [Eqs.(31) and (32)] with $\mu = \frac{1}{r}$, one finds

$$h^{(0)} \simeq \frac{6G^2M\sigma^4}{\pi}(\xi - \tfrac{1}{6})^2 r\ln\frac{r}{r_0}, \quad (63)$$

which coincides with the logarithmic term of result Eq. (61).

V. CONCLUSIONS

Let us summarize the new results contained in this work. We have obtained the in-in effective equations for an arbitrary gravitational field that include the back reaction produced by a quantum scalar field of mass m. The equations are nonlocal, covariant and valid under the assumption $\nabla\nabla\mathcal{R} \gg \mathcal{R}^2$. In the limit $m^2 \gg -\Box$, the equations become local and reproduce the Schwinger-DeWitt expansion. In the opposite limit, $m^2 \ll -\Box$, the presence of nonlocal kernels of the form $\ln(-\frac{\Box}{\mu^2})$ made it possible to read the scaling behavior of the gravitational constants α and β under the rescaling of the metric. This scaling coincides with the renormalization group predictions. This is also the case for the s scaling of the Newton constant, but only for minimal coupling (this fact has been pointed out in Ref. [9]).

Using the in-in equations we computed the quantum corrections to the Newtonian potential. This is our main result. We have found two types of corrections: short range corrections that decay faster than $\frac{1}{r}$ and corrections proportional to $\frac{1}{r}\ln\frac{r}{r_0}$, which we recognized as the scaling of the Newton constant. This scaling coincides with the renormalization group prediction only for minimal and conformal coupling. For other couplings, while the μ dependence of G is proportional to $(\xi - \frac{1}{6})$, the scaling in the Newtonian potential is proportional to $6(\xi^2 - \frac{1}{36})$. Therefore, the "Wilsonian" approach is strictly valid only for $\xi = 0$ and $\xi = \frac{1}{6}$.

One of the main motivations behind the present work was the remark made in Ref. [6] about the possibility of explaining the dark matter problem through the scale dependence of G. In that paper, the running assumed was the one dictated by the renormalization group equations, in a theory of gravity containing R^2 terms. From our results we see that, in the toy model we have considered, the renormalization group behavior is qualitatively but not quantitatively reproduced at the level of the Newtonian potential. However, at present we cannot draw definite conclusions about the R^2 theory, since we have not included the graviton loop in our calculations. We hope to clarify this issue in the future.

Finally, we would like to point out that the covariant effective equations we have found in Sec. III can also be used to analyze the effect of scaling in cosmological situations. For a Robertson-Walker metric with scale factor $a(t)$, we expect local terms of the form $\ln a^2(t)$ to be contained in the kernel $\ln(-\frac{\Box}{\mu^2})$. These local terms may have interesting cosmological and astrophysical consequences, like the generation of a primordial magnetic field during inflation [19]. Work in this direction is in progress.

Note added. While we were writing this article we received a paper by Donoghue [20], where the author calculates the quantum corrections to the Newtonian potential due to the graviton loop. His results are qualitatively the same as ours in the case $m^2 = 0$. This is to be expected, since the physical degrees of freedom of the graviton can be treated as massless scalar fields.

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APPENDIX

In this Appendix we calculate the first quantum correction $h^{(1)}$ that solves the linearized equation of motion, Eq. (55), for the case $m^2 = 0$. The evaluation of the action of the kernel $G(-\frac{\nabla^2}{\mu^2})$ on the Yukawa potential is accomplished using Eqs. (12) and (15):

$$\begin{aligned} G\left(-\frac{\nabla^2}{\mu^2}\right)\frac{e^{-\sigma r}}{r} &= \int d^3x' \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \ln\left(\frac{\mathbf{k}^2}{\mu^2}\right)\frac{e^{-\sigma r'}}{r'} \\ &= \ln\left(\frac{\sigma^2}{\mu^2}\right)\frac{e^{-\sigma r}}{r} - \frac{e^{\sigma r}}{r}\mathrm{Ei}(-\sigma r) - \frac{e^{-\sigma r}}{r}\mathrm{Ei}(\sigma r) \end{aligned} \quad \text{(A1)}$$

where $\mathrm{Ei}(x)$ is the exponential integral function. Taking this expression into account, the equation of motion reads

$$\left(\nabla^2 - \sigma^{-2}\nabla^2\nabla^2\right)h^{(1)} = \sum_{i=1}^{5} f_i(\mathbf{x}) \quad \text{(A2)}$$

where

$$\begin{aligned} f_1(\mathbf{x}) &= 64G^2M\sigma^2\left[\frac{19}{1440} - \frac{5\xi}{24} + \frac{3\xi^2}{4} + \frac{3}{8}\left(\xi - \frac{1}{6}\right)^2 \ln\mu^2\right]\delta^3(\mathbf{x}), \\ f_2(\mathbf{x}) &= 64G^2M\sigma^2\left[\frac{3}{16\pi}\left(\xi - \frac{1}{6}\right)^2\right]\frac{1}{r^3}, \\ f_3(\mathbf{x}) &= 64G^2M\sigma^4\left[-\frac{19}{5760\pi} + \frac{5\xi}{96\pi} - \frac{3\xi^2}{16\pi} + \frac{3}{16\pi}\left(\xi - \frac{1}{6}\right)^2 \ln\frac{\sigma}{\mu}\right]\frac{e^{-\sigma r}}{r}, \\ f_4(\mathbf{x}) &= 64G^2M\sigma^4\left[-\frac{3}{32\pi}\left(\xi - \frac{1}{6}\right)^2\right]\frac{e^{\sigma r}}{r}\mathrm{Ei}(-\sigma r), \\ f_5(\mathbf{x}) &= 64G^2M\sigma^4\left[-\frac{3}{32\pi}\left(\xi - \frac{1}{6}\right)^2\right]\frac{e^{-\sigma r}}{r}\mathrm{Ei}(\sigma r). \end{aligned} \quad \text{(A3)}$$

Being a linear equation, we propose a time-independent, spherically symmetric solution of the form

$$h^{(1)}(\mathbf{x}) = \sum_{i=1}^{5} h_i^{(1)}(r) \quad \text{(A4)}$$

where each $h_i^{(1)}(r)$ is the solution corresponding to the source $f_i(\mathbf{x})$.

Let us denote by $\mathcal{G}(\mathbf{x} - \mathbf{x}')$ the Green function of the operator $\nabla^2 - \sigma^{-2}\nabla^2\nabla^2$ [obviously $\mathcal{G}(\mathbf{x})$ is proportional to $h^{(0)}(\mathbf{x})$]. The solutions $h_i^{(1)}(\mathbf{x})$ are then given by

$$h_i^{(1)}(\mathbf{x}) = \int d^3x'\, \mathcal{G}(\mathbf{x} - \mathbf{x}') f_i(\mathbf{x}'). \quad \text{(A5)}$$

The source $f_1(\mathbf{x})$ is proportional to $\delta^3(\mathbf{x})$. Therefore, $h_1^{(1)}$ is proportional to $h^{(0)}$ and can be absorbed into the classical parameters.

For $i = 2$ we obtain

$$\begin{aligned} h_2^{(1)}(r) &= \frac{24}{\pi^2 r}G^2M\sigma^2(\xi - \tfrac{1}{6})^2 \int_0^\infty dt \frac{\ln t}{t(1+t^2)}\sin(\sigma r t) \\ &= \frac{6}{\pi r}G^2M\sigma^2(\xi - \tfrac{1}{6})^2 \int_0^{\sigma r} dz[e^z\mathrm{Ei}(-z) \\ &\quad - e^{-z}\mathrm{Ei}(z)] \end{aligned} \quad \text{(A6)}$$

where the last equality can be proved by taking r derivatives on both sides and using properties of $\mathrm{Ei}(z)$. Having now the exact first quantum correction, one can analyze the limit $\sigma r \to 0$. Using the series expansion for the exponential integral function [21], the quantum correction reduces to

$$h_2^{(1)} = \frac{6G^2M\sigma^4}{\pi}(\xi - \tfrac{1}{6})^2\left[r\ln(\sigma r) + (\gamma - \tfrac{3}{2})r\right] + O(\sigma^6). \quad \text{(A7)}$$

The other sources can be treated in a similar way. However, as they are all proportional to σ^4 [see Eq. (A3)], the new solutions $h_i^{(1)}$, $i = 3, 4, 5$ are of order σ^6. Therefore,

$$h^{(1)}(\mathbf{x}) = h_2^{(1)}(\mathbf{x}) + O(\sigma^6). \quad \text{(A8)}$$

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